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LETTER TO THE EDITOR

The conformal anomaly and the renormalisation group

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Abstract. The conformal anomaly for an interacting field theory in curved space–time is derived in a simple manner using the renormalisation group.

In this letter, a derivation of the conformal trace anomaly of an interacting field theory in curved space–time is presented, based on the use of the renormalisation group. A discussion of the connection between the conformal trace anomaly and the renormalisation group has recently been given by Brown and Collins (1980). The treatment given below is, however, somewhat simpler than that given by Brown and Collins since the conformal anomaly is obtained without recourse to operator product expansions. The price paid for this simplicity is that the anomaly is determined only up to an arbitrary total divergence.

Consider a scalar field, ϕ, in an n-dimensional space–time with metric $g_{\mu\nu}$. Let the scalar field action be

$$S_{\mathrm{M}}[g_{\mu\nu}, \phi] = \int [-\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}(m_{\mathrm{B}}^2 + \xi_{\mathrm{B}}R)\phi^2 - \tfrac{1}{4}\lambda_{\mathrm{B}}\phi^4]\sqrt{g}\,\mathrm{d}^n x \tag{1}$$

where the subscript B denotes that the coupling constants are bare. The effective action for the matter fields, $W[g_{\mu\nu}]$, is formally given by the path integral

$$\exp(\mathrm{i}W[g_{\mu\nu}]) = \int \exp(\mathrm{i}S_{\mathrm{M}}[g_{\mu\nu}, \phi])[\mathrm{d}\phi] \tag{2}$$

and the semiclassical theory is described by the total action

$$S[g_{\mu\nu}] = S_{\mathrm{G}}[g_{\mu\nu}] + W[g_{\mu\nu}] \tag{3}$$

where

$$S_{\mathrm{G}}[g_{\mu\nu}] = \int [\Lambda_{0\mathrm{B}} + \Lambda_{1\mathrm{B}}R + \alpha_{1\mathrm{B}}R^2 + \alpha_{2\mathrm{B}}R^{\alpha\beta}R_{\alpha\beta} + \alpha_{3\mathrm{B}}R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}]\sqrt{g}\,\mathrm{d}^n x. \tag{4}$$

In equation (4) it has been necessary to include terms quadratic in the Riemann curvature to renormalise divergences which appear in $W[g_{\mu\nu}]$ as $n \to 4$. The total action in n dimensions has the general form

$$S(\Lambda_{0\mathrm{B}}, \Lambda_{1\mathrm{B}}, \underset{\sim}{\alpha}_{\mathrm{B}}, m_{\mathrm{B}}, \xi_{\mathrm{B}}, \lambda_{\mathrm{B}}, n) = S_{\mathrm{G}}(\Lambda_{0\mathrm{B}}, \Lambda_{1\mathrm{B}}, \underset{\sim}{\alpha}_{\mathrm{B}}, n) + W(m_{\mathrm{B}}, \xi_{\mathrm{B}}, \lambda_{\mathrm{B}}, n). \tag{5}$$

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The bare coupling constants have dimensions

$$\begin{array}{lll} \Lambda_{0\mathrm{B}} \sim [\mathrm{mass}]^{n} & \Lambda_{1\mathrm{B}} \sim [\mathrm{mass}]^{n-2} & \underset{\sim}{\alpha}_{\mathrm{B}} \sim [\mathrm{mass}]^{n-4} \\ m_{\mathrm{B}} \sim [\mathrm{mass}]^{1} & \xi_{\mathrm{B}} \sim [\mathrm{mass}]^{0} & \lambda_{\mathrm{B}} \sim [\mathrm{mass}]^{4-n}. \end{array} \tag{6}$$

To renormalise the theory, introduce dimensionless coupling constants α, ξ and λ and coupling constants Λ_0, Λ_1 and m having dimensions

$$\Lambda_0 \sim [\mathrm{mass}]^4 \qquad \Lambda_1 \sim [\mathrm{mass}]^2 \qquad m \sim [\mathrm{mass}]^1 \tag{7}$$

and a unit of mass μ. By dimensional analysis these coupling constants can all be expressed as a power of μ times a dimensionless function of the dimensionless quantities $\mu^{-1}m_{\mathrm{B}}$, ξ_{B} and $\mu^{n-4}\lambda_{\mathrm{B}}$. They do not depend on the bare gravitational couplings as the gravitational field is unquantised. The renormalised total action, S_{R}, is now defined in n dimensions by

$$S_{\mathrm{R}}(\Lambda_0, \Lambda_1, \underset{\sim}{\alpha}, m, \xi, \lambda, \mu, n) = S(\Lambda_{0\mathrm{B}}, \Lambda_{1\mathrm{B}}, \underset{\sim}{\alpha}_{\mathrm{B}}, m_{\mathrm{B}}, \xi_{\mathrm{B}}, \lambda_{\mathrm{B}}, n). \tag{8}$$

The right-hand side is independent of μ, so differentiating with respect to μ keeping all bare couplings fixed gives

$$\left(\mu\frac{\partial}{\partial\mu} + A_0(\lambda)\frac{\partial}{\partial\Lambda_0} + A_1(\lambda)\frac{\partial}{\partial\Lambda_1} + B_i(\lambda)\frac{\partial}{\partial\alpha_i} + \alpha(\lambda)\frac{\partial}{\partial\xi} + \beta(\lambda)\frac{\partial}{\partial\lambda} - m\gamma(\lambda)\frac{\partial}{\partial m}\right)S_{\mathrm{R}} = 0 \tag{9}$$

where the limit $n \to 4$ has been taken and

$$A_0(\lambda) = \lim_{n\to 4} \mu\frac{\partial\Lambda_0}{\partial\mu} \tag{10}$$

$$A_1(\lambda) = \lim_{n\to 4} \mu\frac{\partial\Lambda_1}{\partial\mu} \tag{11}$$

$$B_i(\lambda) = \lim_{n\to 4} \mu\frac{\partial\alpha_i}{\partial\lambda} \qquad i = 1, 2, 3 \tag{12}$$

$$\alpha(\lambda) = \lim_{n\to 4} \mu\frac{\partial\xi}{\partial\mu} \tag{13}$$

$$\beta(\lambda) = \lim_{n\to 4} \mu\frac{\partial\lambda}{\partial\mu} \tag{14}$$

$$\gamma(\lambda) = \lim_{n\to 4} -\frac{\mu}{m}\frac{\partial m}{\partial\mu} = \mu\frac{\partial m}{\partial\mu}\lg\left(\frac{m}{m_{\mathrm{B}}}\right). \tag{15}$$

The bare coupling constants can be expressed as power series in λ in the standard way

$$\mu^{n-4}\lambda_{\mathrm{B}} = \lambda\left(1 + \sum_{\nu=1}^{\infty}\frac{a_\nu(\lambda)}{(n-4)^\nu}\right) \tag{16}$$

$$m_{\mathrm{B}} = m\left(1 + \sum_{\nu=1}^{\infty}\frac{b_\nu(\lambda)}{(n-4)^\nu}\right) \tag{17}$$

$$\xi_{\mathrm{B}} = \xi\left(1 + \sum_{\nu=1}^{\infty}\frac{d_\nu(\lambda)}{(n-4)^\nu}\right) \tag{18}$$

$$\mu^{-n}\Lambda_{0B}=\mu^{-4}\Lambda_0+\sum_{\nu=1}^{\infty}\frac{r_\nu(\mu^{-1}m,\lambda,\xi)}{(n-4)^\nu} \tag{19}$$

$$\mu^{2-n}\Lambda_{1B}=\mu^{-2}\Lambda_1+\sum_{\nu=1}^{\infty}\frac{s_\nu(\mu^{-1}m,\lambda,\xi)}{(n-4)^\nu} \tag{20}$$

$$\mu^{4-n}\alpha_{iB}=\alpha_i+\sum_{\nu=1}^{\infty}\frac{t_{i\nu}(\mu^{-1}m,\lambda,\xi)}{(n-4)^\nu}. \tag{21}$$

Differentiating (16)–(18) with respect to μ leads to the following results:

$$\mu\,\partial\lambda/\partial\mu=-\lambda^2a_1'(\lambda)+\lambda(n-4) \tag{22}$$

$$\mu\,\partial m/\partial\mu=-m\lambda b_1'(\lambda) \tag{23}$$

$$\mu\,\partial\xi/\partial\mu=-\xi\lambda d_1'(\lambda). \tag{24}$$

Hence one obtains the familiar expressions

$$\alpha(\lambda)=-\xi\lambda d_1'(\lambda) \tag{25}$$

$$\beta(\lambda)=-\lambda^2a_1'(\lambda) \tag{26}$$

$$\gamma(\lambda)=\lambda b_1'(\lambda). \tag{27}$$

An analysis of the general structure of the Feynman graphs from which the coefficients r_ν, s_ν and $t_{i\nu}$ are determined shows that r_ν is proportional to $(m/\mu)^4$, s_ν is proportional to $(m/\mu)^2$ and $t_{i\nu}$ is independent of m/μ (Bunch 1981). Hence differentiation of (19)–(21) with respect to μ gives

$$\mu\frac{\partial\Lambda_0}{\partial\mu}=-\mu^4\left(r_1+\lambda\frac{\partial r_1}{\partial\lambda}\right)+(4-n)\Lambda_0 \tag{28}$$

$$\mu\frac{\partial\Lambda_1}{\partial\mu}=-\mu^2\left(s_1+\lambda\frac{\partial s_1}{\partial\lambda}\right)+(4-n)\Lambda_1 \tag{29}$$

$$\mu\frac{\partial\alpha_i}{\partial\mu}=-\left(t_{i1}+\lambda\frac{\partial t_{i1}}{\partial\lambda}\right)+(4-n)\alpha_i. \tag{30}$$

Therefore

$$A_0(\lambda)=-\mu^4(r_1+\lambda r_1') \tag{31}$$

$$A_1(\lambda)=-\mu^2(s_1+\lambda s_1') \tag{32}$$

$$B_i(\lambda)=-(t_{i1}+\lambda t_{i1}') \tag{33}$$

where the prime denotes differentiation with respect to λ. Note that $A_0(\lambda)$ is actually independent of μ and proportional to m^4 (since r_1 is proportional to $(m/\mu)^4$). Similarly $A_1(\lambda)$ is proportional to m^2 and $B_i(\lambda)$ is independent of m and μ. In addition to (28)–(30), recursion relations are obtained which relate the coefficients of the multiple poles in (19)–(21) to r_1, s_1, t_{i1}, $\alpha(\lambda)$, $\beta(\lambda)$ and $\gamma(\lambda)$. To obtain the conformal anomaly from (9) it is convenient to separate S_R into a gravitational and a matter part

$$S_R(\Lambda_0,\Lambda_1,\underset{\sim}{\alpha},m,\lambda,\xi,\mu)=S_G(\Lambda_0,\Lambda_1,\underset{\sim}{\alpha})+W_R(m,\lambda,\xi,\mu) \tag{34}$$

where S_G is given by (4) with bare couplings replaced by renormalised ones. Equation (34) is just the definition of W_R, the renormalised effective action for the matter fields.

Using (34), equation (9) becomes

$$\left(\mu\frac{\partial}{\partial\mu}+\alpha\frac{\partial}{\partial\xi}+\beta\frac{\partial}{\partial\lambda}-m\gamma\frac{\partial}{\partial m}\right)W_{\mathrm{R}}+\left(A_0\frac{\partial}{\partial\Lambda_0}+A_1\frac{\partial}{\partial\Lambda_1}+B_i\frac{\partial}{\partial\alpha_i}\right)S_{\mathrm{G}}=0. \tag{35}$$

Now consider a constant scale transformation of the metric

$$g_{\mu\nu}\to\tilde{g}_{\mu\nu}=\Omega^2 g_{\mu\nu}. \tag{36}$$

This rescales all lengths and so its effect on W_{R} is

$$W_{\mathrm{R}}[\Omega^2 g_{\mu\nu};\, m,\, \xi,\, \lambda,\, \mu]=W_{\mathrm{R}}[g_{\mu\nu};\, \Omega m,\, \xi,\, \lambda,\, \Omega\mu]. \tag{37}$$

Hence

$$\left(\Omega\frac{\partial}{\partial\Omega}-m\frac{\partial}{\partial m}-\mu\frac{\partial}{\partial\mu}\right)\tilde{W}_{\mathrm{R}}=0 \tag{38}$$

where the tilde indicates that the scaled metric $\tilde{g}_{\mu\nu}$ is used. Combining (35) and (38) gives

$$\left(\Omega\frac{\partial}{\partial\Omega}+\alpha\frac{\partial}{\partial\xi}+\beta\frac{\partial}{\partial\lambda}-m(1+\gamma)\frac{\partial}{\partial m}\right)\tilde{W}_{\mathrm{R}}+\left(A_0\frac{\partial}{\partial\Lambda_0}+A_1\frac{\partial}{\partial\Lambda_1}+B_i\frac{\partial}{\partial\alpha_i}\right)\tilde{S}_{\mathrm{G}}=0. \tag{39}$$

Hence the trace of the energy momentum tensor is determined up to a total divergence by

$$\int\langle T^{\alpha}_{\alpha}\rangle_{\mathrm{REN}}\sqrt{g}\,\mathrm{d}^4x\equiv\Omega\frac{\partial\tilde{W}_{\mathrm{R}}}{\partial\Omega}\bigg|_{\Omega=1}$$

$$=\left(m(1+\gamma)\frac{\partial}{\partial m}-\alpha\frac{\partial}{\partial\xi}-\beta\frac{\partial}{\partial\lambda}\right)W_{\mathrm{R}}-\left(A_0\frac{\partial}{\partial\Lambda_0}+A_1\frac{\partial}{\partial\Lambda_1}+B_i\frac{\partial}{\partial\alpha_i}\right)S_{\mathrm{G}}. \tag{40}$$

When $m=0$ the anomalous trace is

$$\int\langle T^{\alpha}_{\alpha}\rangle_{\mathrm{REN}}\sqrt{g}\,\mathrm{d}^4x=\left(-\alpha(\lambda)\frac{\partial}{\partial\xi}-\beta(\lambda)\frac{\partial}{\partial\lambda}\right)W_{\mathrm{R}}-B_i(\lambda)\frac{\partial S_{\mathrm{G}}}{\partial\alpha_i} \tag{41}$$

$$=\left(\xi\lambda d_1'\frac{\partial}{\partial\xi}+\lambda^2 a_1'\frac{\partial}{\partial\lambda}\right)W_{\mathrm{R}}+(t_{i1}+\lambda t_{i1}')\frac{\partial S_{\mathrm{G}}}{\partial\alpha_i}. \tag{42}$$

When $m\neq0$ the contributions proportional to $A_0(\lambda)$, $A_1(\lambda)$ and $\gamma(\lambda)$ may also be regarded as anomalous.

References

Brown L S and Collins J C 1980 *Ann. Phys., NY* **130** 215
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